3.5 Correspondence between Lie subalgebras and Lie subgroups.

We would like to address the following question:

<u>Q</u>: Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Is there a subgroup of G "naturally associated" to $\mathfrak{h}$?

Exercise 3.71

Understand why we already answered to this question in the case when $\mathfrak{h}$ is 1-dimensional in the previous section.

We will need some additional terminology and some tool from differential geometry.

Definition 3.72 [Immersed - embedded submanifolds].

Let $\varphi : N \longrightarrow M$ be a smooth map between smooth manifolds. Then:

1) We say that φ is an immersion if $D_p\varphi$.

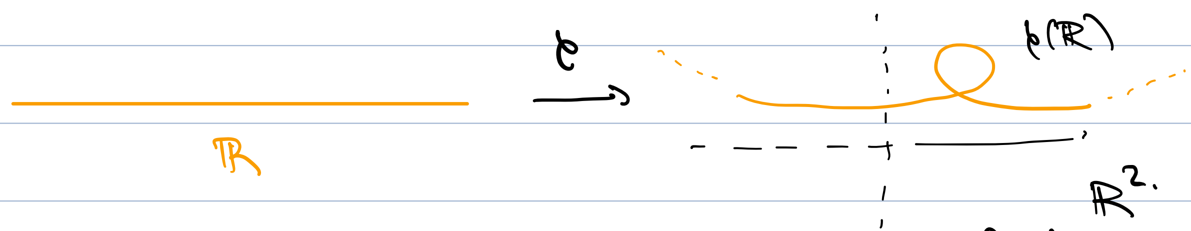
is <u>injective</u> $\forall\, p \in N$.

2) We say that $\iota(N)$ is an <u>immersed</u> <u>submanifold</u> of M if ι is a <u>one-to-one</u> <u>immersion</u>.

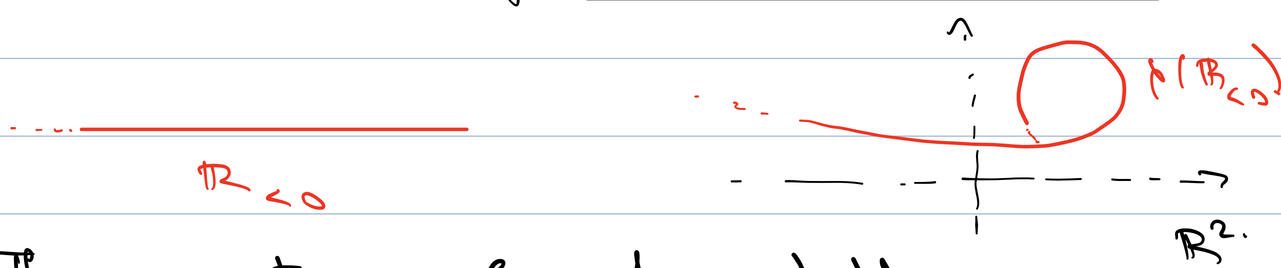
3) If ι is a <u>one-to-one immersion</u> that is also a <u>homeomorphism onto its image</u> then we say that ι is <u>an embedding</u> and $\iota(M)$ is an <u>embedded submanifold</u>.

Example 3.73

This is a example of an <u>immersion</u>:



This is an example of <u>immersed submanifold</u>:



They <u>are not</u> examples of <u>embeddings</u>.

Exercise 3.74.

Understand in which sense an <u>embedded submanifold</u>

is the same as a <u>regular submanifold</u> (see Definition 3.10).

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We come back to our question and discuss an example

Example 3.75

Let $G = T^2 = S^1 \times S^1$. Then the Lie algebra $\mathfrak{g}$ of G coincides with the Lie algebra of $\mathbb{R}^2$, i.e, $\mathbb{R}^2$ with trivial bracket.

1) If $\mathfrak{h} = \mathbb{R} \simeq \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and if we let $i : S^1 \longrightarrow T^2$ be defined by $i(\theta) = (0, \theta)$ then $i(S^1) = \{0\} \times S^1$ is a regular submanifold and a subgroup and $\mathrm{Lie}(i(S^1)) = \mathfrak{h}$.

2) If $\mathfrak{h} = \{(x, y) \in \mathbb{R}^2 : y = \sqrt{2}x\}$ then we can let $\phi : \mathbb{R} \longrightarrow T^2$ be defined as $\phi(t) := (e^{it}, e^{i\sqrt{2}t})$. Then

ϕ is a one-to-one immersion and $\phi(\mathbb{R}) =: H$ is a subgroup of T^2 and an immersed submanifold with. Lie algebra h. However H is not a regular submanifold.

—— o ——

The effect of the above Example 3.75 is that in general we cannot expect the Lie subgroup associated to a Lie subalgebra to be embedded.

Let's introduce the relevant definition:

$\boxed{\text{Definition 3.76}}$ [Lie subgroup]

Let G be a Lie group. We say that (H, ϕ) is a Lie subgroup of G if

1) H is a Lie group;
2) $\phi : H \longrightarrow G$ is an injective Lie group homomorphism;
3). $\phi(H)$ is an immersed submanifold, i.e., ϕ is a one-to-one immersion.

the key theorem concerning the correspondence between Lie sub-algebras and Lie-subgroups is the following:

Theorem 3.77

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{h}} \subseteq \mathfrak{g}$ be a <u>Lie subalgebra</u>. Then <u>there is</u> <u>a unique connected Lie subgroup</u> (H, φ) of G such that $D_e\varphi(\mathfrak{h}) = \tilde{\mathfrak{h}}$, where $\mathfrak{h}$ is the Lie algebra of H.

Remark 3.78

<u>Uniqueness</u> in the statement of Theorem 3.77 is understood in the following sense. We say that two Lie subgroups (H, φ) and (H_1, φ_1) of G <u>are equivalent</u> if there is a <u>Lie group isomorphism</u> $\alpha : H \to H_1$ such that $\varphi_1 \circ \alpha = \varphi$.

$$\begin{tikzcd} H \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & G \\ & H_1 \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

Uniqueness has to be understood <u>up to</u> <u>equivalence</u>.

———— ○ ————

The proof of Theorem 3.77 relies on

an important result in differential geometry. We start by discussing some motivation.

Let M be a smooth manifold and $X \in \mathrm{Vect}^\infty(M)$ such that $X_p \neq 0 \quad \forall\, p \in M$. We assume that it is complete for the sake of simplicity.

By Theorem 3.46 for every $p \in M$, we can find an integral curve $\gamma_p : \mathbb{R} \longrightarrow M$ of X, i.e, $\gamma_p(0) = p$ and $\gamma_p'(t) = X_{\gamma_p(t)}$ for each $t \in \mathbb{R}$.

In particular $\gamma_p : \mathbb{R} \longrightarrow M$ is an immersion and the tangent space of $\gamma_p(\mathbb{R})$ in M is the span of X at each point.

We would like to generalize this construction to the "higher dimensional" case.

For instance, suppose that we have $X_1, X_2 \in \mathrm{Vect}^\infty(M)$, then we would like to find a two-dimensional submanifold whose tangent space is spanned by

X_1 and X_2 at each point.

Exercise 3.79

Understand why such a submanifold cannot exist in general without further conditions.

Definition 3.80 [Distributions]

Let M be a smooth manifold of dimension $n+k$.

1). For each $p \in M$ consider an n-dimensional subspace $\mathcal{D}_p \subset T_pM$. Suppose that in a neighborhood U of any point $p \in M$, there are n linearly independent smooth vector fields $X_1, \dots, X_n$ that give a basis of $\mathcal{D}_q$ $\forall q \in U$. Then we say that $\mathcal{D}$ is a <u>smooth distribution of dimension n</u> and $X_1, \dots, X_n$ is a <u>local basis</u> of $\mathcal{D}$.

2). We say that a distribution <u>is involutive</u> if there exists a <u>local basis $X_1, \dots, X_n$</u> of $\mathcal{D}$ near to each point such that

$$[X_i, X_j] \in \mathcal{D} \quad \forall\, 1 \le i \le j \le n.$$

3) If $\mathcal{D}$ is a <u>smooth distribution</u> on M and $\varphi : N \longrightarrow M$ is a <u>one-to-one immersion</u> we say that $\varphi(N)$ is an <u>integral submanifold</u> of $\mathcal{D}$ if $D_p\varphi(T_pN) = \mathcal{D}_{\varphi(p)} \quad \forall\, p \in N$.

4). We say that a distribution $\mathcal{D}$ on M is <u>completely integrable</u> if it admits an integral submanifold through each point.

Example 3.8.1

- If $M = \mathbb{R}^n \times \mathbb{R}^k$ and $X_i = \dfrac{\partial}{\partial x_i}$ for $i = 1, \dots n$ then $\mathcal{D} = \operatorname{Span}\{X_1, \dots X_n\}$ <u>is an involutive distribution</u>.

- If $M = \mathbb{R}^3$ then the distribution $\mathcal{D} = \operatorname{Span}\left\{ \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} + x\dfrac{\partial}{\partial z} \right\}$ <u>is not involutive</u>.

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The first key theorem regarding distributions is the following:

Theorem 3.82 [Frobenius]

A smooth distribution is completely integrable if and only if it is involutive.

Remark 3.83

If the distribution is 1-dimensional then it is automatically involutive since $[X, X] = 0$ for any $X \in \mathrm{Vect}^{\to}(M)$. Then Theorem 3.82 is a consequence of Theorem 3.46.

Definition 3.84 [Maximal integral submanifold]

A maximal integral submanifold N of an involutive distribution $\mathcal{D}$ on a manifold M is a connected integral submanifold of $\mathcal{D}$ whose image in M is not a proper subset of the image of any other connected integral submanifold of $\mathcal{D}$.

Cf with the notion of maximal integral curve

Theorem 3.85

Given an involutive distribution on a manifold M and a point $p \in M$ there exists a unique maximal integral submanifold through p.

———— o ————

For the proofs of Theorem 3.82 and Theorem 3.85 see for instance [Warner, Chapter 1].

We will need one last technical tool in the proof namely:

$\boxed{\text{Proposition 3.86}}$

Let M be a smooth manifold and $\mathcal{D}$ be an involutive distribution on M. Let N be a maximal integral submanifold. If $f: M' \to M$ is a smooth map, where M' is another smooth manifold and $f(M') \subseteq \varphi(N)$ then $\varphi^{-1} \circ f : M' \to N$ is a smooth map.

See [Warner, Theorem 1.62] for a proof.

Proof of Theorem 3.77

We define a distribution $\mathcal{D}$ on M by setting
$$\mathcal{D}_g = \{ X^L_g : X \in \tilde{\mathfrak{h}} \}.$$

We claim that $\mathcal{D}$ is a smooth distribution. Indeed if $X_1, \dots X_d \in \tilde{\mathfrak{h}}$ form a basis of $\tilde{\mathfrak{h}}$ then $\mathcal{D}$ is spanned globally by the smooth vector fields $X_1^L, \dots X_d^L$.

Moreover, $\mathcal{D}$ is involutive, because $\tilde{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}$. Indeed if X and Y are vector fields lying in $\mathcal{D}$ then there are smooth functions a_i, b_i such that $X = \sum_i a_i X_i^L$ and $Y = \sum_i b_i X_i^L$.

Hence
$$[X,Y] = \sum_{i,j} \{ a_i b_j [X_i^L, X_j^L] + a_i X_i^L(b_j) X_j - b_j X_j(a_i) X_i \}$$
belongs to $\mathcal{D}$ since it is a linear combination of elements of $\mathcal{D}$ at each point.

Let (H, e) a maximal connected integral submanifold of $\mathcal{D}$ through e, as given by

Theorem 3.85. Let $g \in \varphi(H)$. Since $\mathcal{D}$ is invariant under left translations also $(H, L_{g^{-1}} \circ \varphi)$ is an integral submanifold of $\mathcal{D}$ through e. By maximality $L_{g^{-1}} \circ \varphi(H) \subseteq \varphi(H)$.

Therefore if $g \in \varphi(H)$ and $g' \in \varphi(H)$ then also $g^{-1}g' \in \varphi(H)$. Therefore $\varphi(H) < G$ and we can induce a group structure on H in such a way that $\varphi : H \longrightarrow G$ is a homomorphism.

To conclude we need to check that H is a Lie group, i.e, the original smooth structure and the operations defined by pulling back those of $\varphi(H) < G$ are compatible.

Note that the map $\beta : H \times H \longrightarrow G$
$$(g, g') \longmapsto \varphi(g)\varphi(g')^{-1}$$
is smooth by smoothness of φ and of the operations in G. Moreover it has clearly values in $\varphi(H)$.

Then, if we denote
$$\alpha : H \times H \longrightarrow H, \quad g, g' \longmapsto g(g')^{-1}$$
we have a commutative diagram

$$\begin{tikzcd}
H \times H \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & G \\
& H \arrow[u, "\varphi"']
\end{tikzcd}$$

and by Proposition 3.86 we infer that α is smooth, i.e., H is a Lie group. Therefore (H, φ) is a Lie subgroup of G and by construction $D_e\varphi(\mathfrak{h}) = \tilde{\mathfrak{h}}$.

The proof of the uniqueness part of the statement is based on similar ideas and left to the reader. For the detailed argument see [Warner, Theorem 3.19]. □

In general, φ is not an embedding, see Example 3.75.

$\boxed{\text{Theorem 3.87}}$

Let (H, φ) be a Lie subgroup of a Lie group G. Then φ is an embedding iff $\varphi(H)$ is closed in G.

For a proof you can look at [Warner, Theorem 3.21].

We saw that a smooth Lie group homomorphism naturally induces a Lie algebra homomorphism in Proposition 3.39. We would like to understand whether some converse holds.

$\boxed{\text{Example 3.88}}$

Let $p : \mathbb{R} \to S^1$ be defined as $p(t) := e^{it}$. Then $D_0 p : \mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(S^1)$ is a Lie algebra isomorphism, and so is $(D_0 p)^{-1} : \mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$. However $(D_0 p)^{-1}$ cannot be the derivative of any homomorphism $\varphi : S^1 \to \mathbb{R}$. Indeed the image of any such homomorphism is a compact subgroup of $\mathbb{R}$ hence trivial.

Nevertheless $(D_0 p)^{-1}$ comes from a local homomorphism, namely the local inverse of p.

In Chapter 2 we gave the definition of local homomorphism for topological groups. For Lie groups the definition has to be adjusted so as to require additionally smoothness.

Then we have the following

$\boxed{\text{Theorem 3.89}}$

1) If G and H are Lie groups and $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism of the corresponding Lie algebras then there exists a local Lie group homomorphism $\varphi : U \longrightarrow H$ such that $D_e \varphi = \pi$.

2). If π is a Lie algebra isomorphism then φ is a local isomorphism.

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Part 2) follows from part 1) via the following.

Lemma 3.90

If $\varphi : U \to H$ is a local Lie group hom. such that $D_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism then φ is a local isomorphism.

Proof

Since $D_e\varphi$ is an isomorphism by the Inverse function theorem there exist a neighborhood U' of $e_G \in G$ and a neighborhood V of $e_H \in H$ such that $\varphi : U' \to V$ is a diffeomorphism.
Then φ is a local isomorphism on U'. $\square$

Proof of Theorem 3.89

We claim that $\mathrm{Graph}(\pi) := \{(X, \pi(X)) : X \in \mathfrak{g}\}$ $\subset \mathfrak{g} \times \mathfrak{h}$ is a Lie subalgebra.
Indeed if $X, Y \in \mathfrak{g}$ then:

$$[(X, \pi(X)), (Y, \pi(Y))] = ([X,Y], [\pi(X), \pi(Y)])$$
$$= ([X,Y], \pi([X,Y])).$$

By Theorem 3.77 there exists an a Lie subgroup $K < G \times H$ such that $\mathrm{Lie}(K) = \mathrm{Graph}(\pi)$ (with a slight abuse of notation).

Therefore we have

$$\mathrm{Graph}(\pi) \hookrightarrow \mathfrak{g} \times \mathfrak{h} \xrightarrow{D_e pr_G} \mathfrak{g}$$

and

$$K \hookrightarrow G \times H \xrightarrow{pr_G} G.$$

By construction $pr_G|_K : K \to G$ is a Lie group homomorphism and its derivative at e $D_e pr_G|_K = pr_{\mathfrak{g}}|_{\mathrm{Graph}(\pi)} : \mathrm{Graph}(\pi) \to \mathfrak{g}$ is a Lie algebra isomorphism.

Hence by Lemma 3.90 $pr_G|_K$ is a local isomorphism, that is, there exist neighb $e_K \in W \subset K$ and $e_G \in V \subset G$ such that $pr_G|_K : W \to V$ is a diffeomorphism

We can consider then $(pr_G|_W)^{-1} : V \to W$ whose derivative $D_e (pr_G|_W)^{-1} : \mathfrak{g} \to \mathrm{Graph}(\pi)$ is the map $X \longmapsto (X, \pi(X))$ by construction

We consider then the homomorphism

$pr_H : G \times H \longrightarrow H$ and its derivative.

$D_e pr_H = pr_h : \mathfrak{g} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ which is a Lie algebra homomorphism. Then:

$$pr_H \circ (pr_G|_W)^{-1} : V \longrightarrow H$$

is the required local homomorphism since:

$$D_{e_G}(pr_H \circ (pr_G|_W)^{-1})(X) =$$

$$= D_{e_G} pr_H \circ D_{e_G}(pr_G|_W)^{-1}(X)$$

$$= D_{e_G} pr_H \,(X, \pi(X))$$

$$= \pi(X) \qquad \forall\, X \in \mathfrak{g}. \qquad \square$$

We state without proof a deep result due to Ado:

$\boxed{\text{Theorem 3.9.1}}$

Any finite dimensional real Lie algebra $\mathfrak{g}$

is isomorphic to a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ for some n.

The combination of Theorem 3.91, Theorem 3.89, and Theorem 3.77 gives:

Corollary 3.92.

Any Lie group is locally isomorphic to a subgroup of $GL(n,\mathbb{R})$ for some n.

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Moreover we have the following:

Corollary 3.93

1) If G is a connected Lie group with Lie algebra $\mathfrak{g}$ then there is a simply connected Lie group $\tilde{G}$ with Lie algebra isomorphic to $\mathfrak{g}$.

2) If two simply connected Lie groups have isomorphic Lie algebras then they are isomorphic.

Proof

Recall Exercise 7 Sheet 1, and check that $\tilde{G}$ can be endowed with a unique smooth structure such that the operations lifted from G make it into a Lie group and $p : \tilde{G} \to G$ is a smooth homomorphism whose kernel is a discrete subgroup of $\tilde{G}$.
Then $D_e p : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a Lie algebra isomorphism. This completes the proof of 1).

In order to prove 2) let G_1, G_2 be simply connected Lie groups with isomorphic Lie algebras $\mathfrak{g}_1 \cong \mathfrak{g}_2$.
By Theorem 3.31 there exists a local isomorphism $p : U \to G_2$ where U is an open neigh of e_{G_1}. By Theorem 2.37 p extends to a homomorphism $G_1 \to G_2$. It is possible to check that the extension is a smooth covering map (Exercise). Since G_2 is simply connected, the extension

is an isomorphism. $\square$

We end this section with a brief discussion about Cartan's theorem on closed subgroups of Lie groups and a few related results.

Theorem 3.94

Let G be a Lie group and H be a closed subgroup of G. Then H has a unique smooth structure compatible with the induced topology which makes it into a Lie subgroup of G.

Proof sketch.

The main idea of the proof is to show that

$$\mathfrak{a} := \{X \in \mathfrak{g} : \exp(tX) \in H \ \forall t \in \mathbb{R}\}$$

is a subspace of the Lie algebra $\mathfrak{g}$ so then we can apply the following.

Lemma 3.95

Let H be a subgroup of a Lie group G.
Let $\mathfrak{a}$ be a subspace of the Lie algebra $\mathfrak{g}$
Let $0 \in U_0 \subset \mathfrak{g}$ and $e \in V_e \subset G$
be open neighbs such that $\exp : U_0 \to V_e$
is a diffeo. Suppose that

$$\exp (U_0 \cap \mathfrak{a}) = V_e \cap H$$

Then

1) H with the induced topology is a Lie subgroup of G.
2) $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$.
3) $\mathfrak{a}$ is the Lie algebra of H.

In particular, a step of the proof of Theorem 3.34 shows the following.

Corollary 3.36

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H < G$ be a closed subgroup.
Then its Lie algebra is given by

$$\mathrm{Lie}(H) = \{ X \in \mathfrak{g} : \exp_G (tX) \in H \ \forall t \in \mathbb{R} \}$$

Among the consequences of Corollary 3.96 there is the following

Corollary 3.97
Let $\pi : G_1 \longrightarrow G_2$ be a smooth homo. with derivative $D\pi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$. Then $\mathrm{Lie}(\ker \pi) = \ker D\pi$.

Proof
By Corollary 3.96 $\mathrm{Lie}(\ker \pi) =$
$= \{ X \in \mathfrak{g} : \pi(\exp_{G_1}(tX)) = e \ \forall t \in \mathbb{R} \}$

By Proposition 3.59
$$\pi(\exp_{G_1}(tX)) = \exp_{G_2}(t D\pi(X))$$
hence.
$$\mathrm{Lie}(\ker \pi) = \{ X \in \mathfrak{g} : \exp_{G_2}(t D\pi(X)) = e \ \forall t \in \mathbb{R} \}.$$
$$= \{ X \in \mathfrak{g} : D\pi(X) = 0 \}.$$
$$= \ker D\pi .$$ $\square$

Corollary 3.61

We end this section with a definition

that will turn out to be useful later.

Definition 3.98 [Ideal]

An ideal $\mathfrak{h}$ in a K-Lie algebra $\mathfrak{g}$ is a vector subspace such that:

$$[x, y] \in \mathfrak{h} \quad \forall x \in \mathfrak{g} \; \forall y \in \mathfrak{h}.$$

Note: the Lie bracket on $\mathfrak{g}$ descends to $\mathfrak{g}/\mathfrak{h}$ to define a Lie algebra structure such that $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ is a Lie algebra homomorphism with $\operatorname{Ker} \pi = \mathfrak{h}$.

Conversely, if $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism then $\operatorname{Ker} \varphi$ is an ideal in $\mathfrak{g}_1$ (Exercise).